Chaos, Self-Similarity, Musical Phrase and Form

Gerald Bennett

The idea of chaos is aesthetically strangely satisfying. Chaos represents the antithesis of artistic production, but it also marks the edge of an abyss along which art often wanders, letting the fumes from below cast a lightly corrosive coat over the order the artist has worked so hard to create. Art can overcome, and may even to a certain degree thrive on, chaos in the physical, everyday world, but real chaos, chaos of the mind and soul, is horribly destructive. Marina Tsvetayeva, Sylvia Plath, Bernd Alois Zimmermann and Paul Celan are but a few of those in our century whose art was born at that edge between order and chaos but whom the chaos finally overwhelmed.

The art born on this dangerous abyss has an authenticity which seems to condemn to banality that arising in safer regions. Much of the important art, poetry and music of our century comes from this border where chaos, in the elemental, existential sense of the word, meets the will to order. We all carry echoes of primal chaos in us; these echoes allow us to understand the wonder of art which creates safe havens (for that is the function of structure) from the chaos which is always threatening to impinge upon us .

I shall not write further here about this sense of chaos in music or other art, but I set these lines at the beginning of my essay in order to characterize the fascination which the idea of chaos (but not the reality) exerts upon us all. We are all the more attracted by the idea of a mathematical Theory of Chaos and of formulas which permit us to seem to create chaos without actually venturing out to the rim of darkness and destruction.

I shall examine briefly some of the simple forms of "artificial" chaos and in particular two aspects of chaotic systems: self-similarity and scaling invariance. I shall speak of a few examples of both in music of the past, and I shall reflect on their appropriateness in musical composition. Finally, in an appendix I shall include a brief and incomplete list of source materials which may be of help to those who want to explore Chaos in a more detailled way. I have always been, and I remain, sceptical about the deep interest of Chaos Theory for musical composition, for I distrust the completely deterministic mechanisms used to simulate chaos. Nonetheless, preparing this short essay gave me the chance to explore deterministic chaos more carefully than I had hitherto done, and I have tried to do so with as open a mind as possible. This text is the record of my exploration and discovery of chaos, the logistic difference function, the Mandelbrot Set and much else.

One of the simplest mathematical expressions of chaos is the so-called logistic difference equation, first formulated slightly differently by the Belgian sociologist and mathematician Pierre-Franois Verhulst in 1845 to model the growth of populations limited by finite resources. Today we write the function like this [4]:

$$f(x) = px \cdot (1 - x)$$

The variable x can take on values from 0.0 to 1.0 while p (often called the growth factor) goes from 0.0 to 4.0. Figure 1 shows the graph of the function:

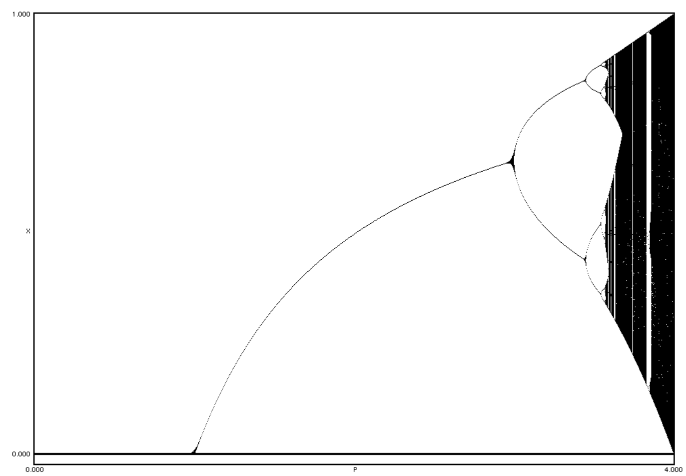

Figure 1. Graph of the function $F(x) = px \cdot (1 - x)$, where p varies from 0 to 4 (x-axis).

For values of p between 0 and 1, the value of the function (the population, of animals, for instance), is 0, For values of p between 1 and 3, the population grows regularly. When p = 3, the function bifurcates (in fact, successive values of the function alternate between the values shown by the two lines). Somewhat later, each line bifurcates again (i.e., the function oscillates between four values). The function continue bifurcating until the behavior of the function is so complex that it appear chaotic. In fact, however, there are fascinating regularities hidden in this apparent chaos.

Figure 2a shows an enlargement of part of Figure 1. Within the chaotic part of the signal, white bands of apparent inactivity are visible. Within the largest of these bands is a small disturbance which, when enlarged, reveals exactly the same structure as the function itself (except for mirror symmetry).

There are infinitely many of these structures. Figures 3 and 4 show two more enlargements.

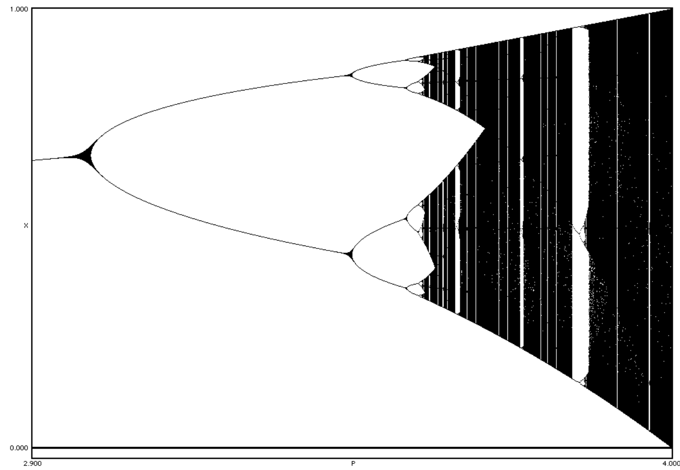

Figure 2. Enlargement of part of Figure 1, beginning shortly before the first bifurcation.

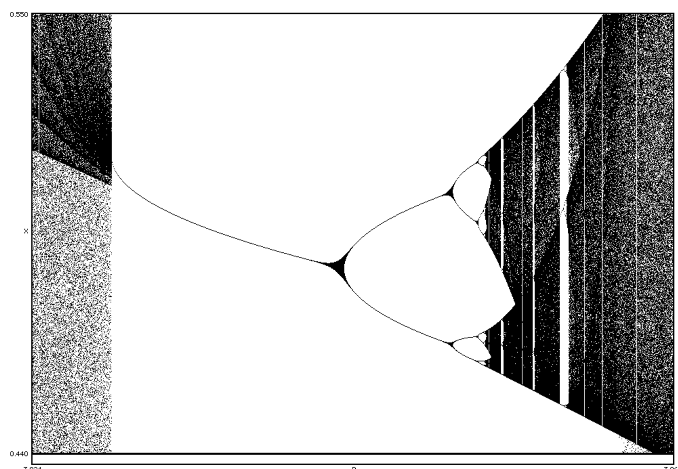

Figure 3. Enlargement of part of the broad white band in the right part of Figure 2. This represents a magnification of about 110x.

The enlargements show two very important characteristics of chaotic systems. The first is self-similarity. Each of the magnified areas first seems to be a small filament in an island of non-activity. But when the area is enlarged, it is seen to have the same structure as the first bifurcations of the function around $p = 3.0$. This self-similarity continues infinitely. The

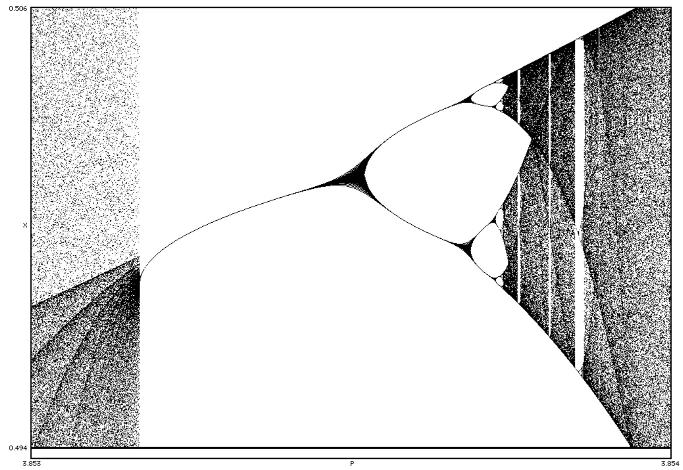

Figure 4. Enlargement of part of the white band in the right part of Figure 3. Magnification on the x-axis of about nearly 6000x.

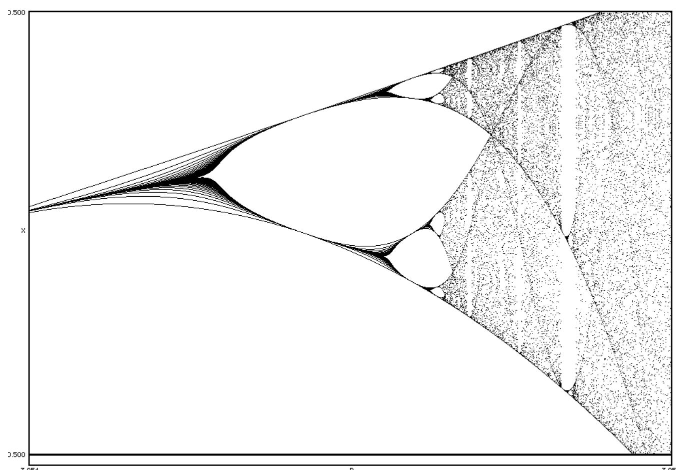

Figure 5. Another enlargement. Magnification on the x-axis of more than 6,500,000x.

second characteristic, scaling invariance, is related to the first. Except for the diminishing density of the points in the Figures 4 and 5 (an artefact of the program used to make the images), there is no way to tell the scale of the graphs.

At bifurcation, the function oscillates between first 2, then 4, then 8, 16, 32, etc. values, until the cycles become so complex as to give the impression of chaos. Figures 6–8 show in detail the behavior of the function near bifurcations.

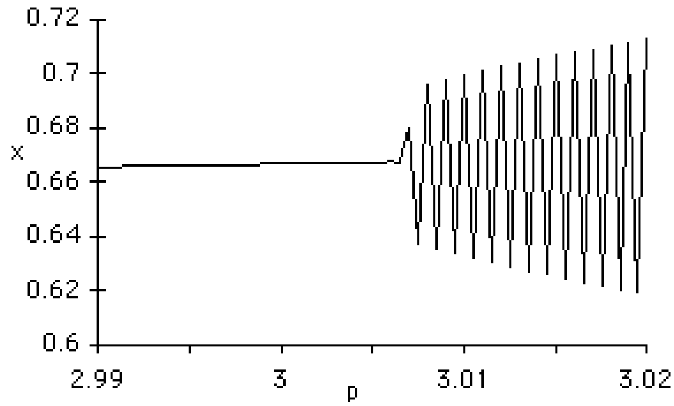

Figure 6. Detail of the logistic difference function as the bifurcation process begins.

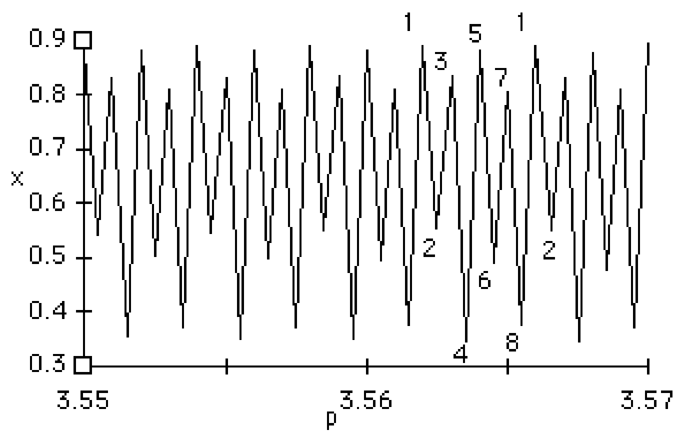

Figure 7. Detail of the logistic difference function with eight evolving values. This corresponds to the last clearly visible set of bifurcations in Figure 2.

Figure 9 shows a detailled view of the spectrum of the function near the passage to apparent chaos, indicating incidentally the presence of oscillating components in the function before actual bifurcation takes place.

It is easy to imagine musical uses of the logistic difference function. The envelope could be used to derive pitch or amplitude, to drive a filter or describe the formal evolution of a section or an entire piece. The American composer Gary Lee Nelson describes how the

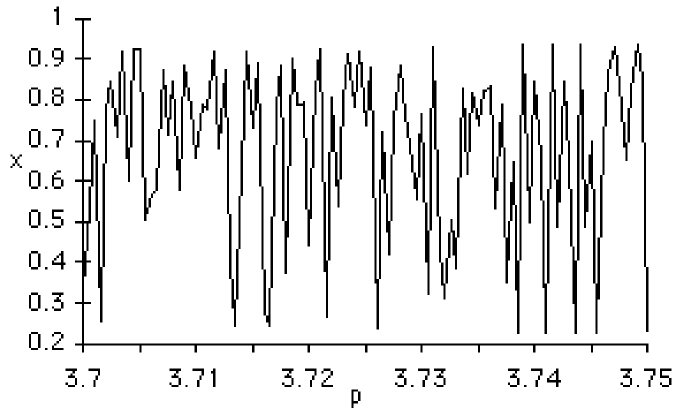

Figure 8. The logistic difference function showing apparently chaotic behavior.

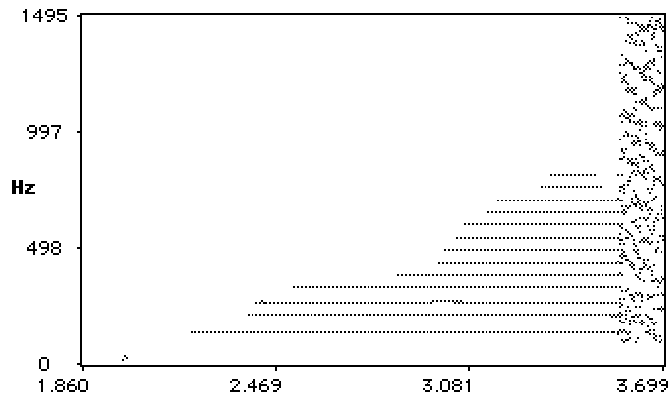

Figure 9. Detail of the spectrum of the logistic difference function, showing five partials in the signal before the first bifurcation at $p = 3.0$*, then eight more partials after* $p = 3.0$*. (Cf. Figure 7.) The passage to "chaos" takes place at about* $p = 3.57$*.*

functions envelope determines the form of his composition *The Voyage of the Golah Iota* and how the function drives a granular synthesis routine to produce sound.

But one can also imagine using the signal itself for synthesis. By using very small stretches of the signal, one can obtain signals of infinitely varied timbral quality. Filters controlled by the function could sweep over the rich timbres generated by the function as signal. Changes in the filters bandwidths, amplitudes, density of sound, and many other compositional aspects could all be controlled by the same function, thus assuring self-similarity over many dimensions of the composition. Three program examples for calculating and using the logistic difference equation will be found in the appendix. The first is a very simple C program to calculate the numbers of the function, the other two are examples of Csound instruments, one for calculating the function directly as a sound file, the other for using the function to control the pitch of an oscillator.

Another familiar type of system which shows both remarkable self-similarity and scaling invariance is the Mandelbrot Set, named for the mathematician Benoit B. Mandelbrot, who invented the name "fractal" for mathematical entities having fractional dimensionality and whose book *The Fractal Geometry of Nature* [6] has inspired so many musicians and artists to investigate self-similarity. The Mandelbrot Set is a connected set of points in the complex plane. It can be constructed as follows. Choose a point Z_0 in the complex plane. Calculate:

$$
\begin{aligned}
Z_1 &= Z_0^2 + Z_0 \\
Z_2 &= Z_1^2 + Z_0 \\
Z_3 &= Z_2^2 + Z_0 \\
etc.
\end{aligned}
$$

If the series Z_0, Z_1, Z_2, etc. remains within a distance of 2.0 from the origin (0,0) "forever", then it is in the Mandelbrot Set. However, if for a point the series takes on values greater than 2.0, then that point is not in the set, but rather belongs to one of an infinite number of "dwell bands", corresponding to the number of iterations before the point moved outside the escape radius 2.0. The intrinsic mathematical interest of the Mandelbrot Set may seem small, but when displayed graphically on a computer, these points give rise to strikingly beautiful images, here shown unfortunately only in shades of grey. Figure 10 shows the Mandelbrot Set as the black, cardoid shape in the center of the image. The various shadings of grey around it correspond to the "dwell bands" into which the points outside the set fall. That is, the patterns of color beyond the Mandelbrot Set proper show after how many iterations a point passed beyond the "escape radius". These dwell bands form the basis for color distinction in colored representations.

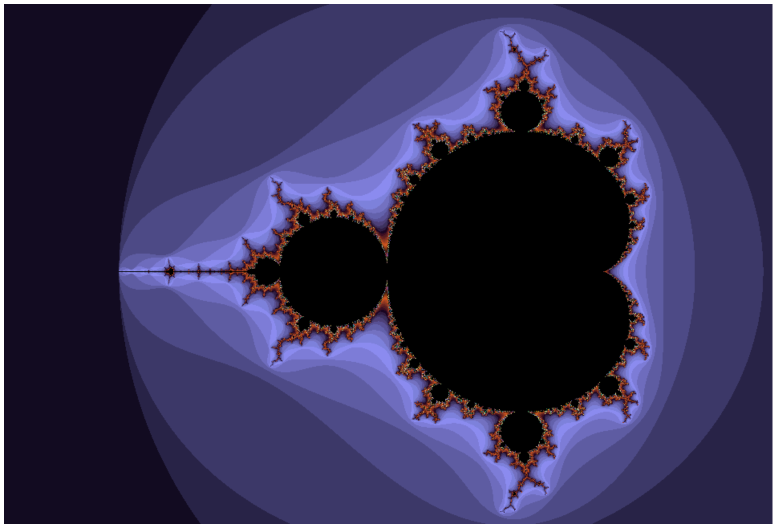

Figure 10. A visualization of the Mandelbrot Set. The Set itself is the black cardoid shape in the middle of the image with a large protuberance to the left and smaller protuberances above and below. The shadings correspond to "dwell bands" into which the points outside the Set fall.

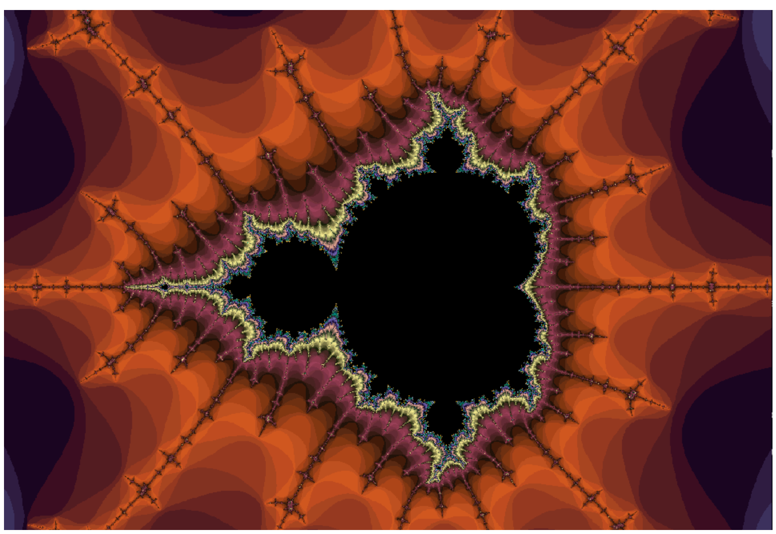

Figure 11. Part of the complex plane of Figure 10 along the horizontal axis to the left of the Set in Figure 10; magnification ca. 200x.

The Mandelbrot Set, or more exactly, the complex plane ordered according to membership or not in the Mandelbrot Set is an entire microcosm, with infinitely many nooks and crannies to explore, many of which yield beautiful visual representations. I must say that while these images are very orderly, I feel a kind of vertigo looking into them and realizing that these delicate structures replicate themselves on an ever smaller scale into infinity. The mechanical implacability of this replication and the cold delicacy of the figures themselves seem to me to reside in that area of experience I consider chaotic in the non-mathematical sense. Georg Cantor, the famous German mathematician who lived from 1845-1918 and whose work has proven so important for Chaos Theory, expressed this feeling: "Eine Menge stelle ich mir vor wie einen Abgrund" (I imagine a set to be an abyss).

It is more difficult to think of musical applications of the Mandelbrot Set, although the idea of a reiterative function whose results go into classes (here the "dwell bands") would seem to me quite amenable to compositional use. I mention the Mandelbrot Set here in such detail because it is the best-known example of how in Chaos Theory mathematics seems to take on an aesthetic significance. I shall speak of this phenomenon later.

The two central characteristics of chaotic systems, self-similarity and scaling invariance, have traditionally been of less importance in music, but we can find examples of both. From many examples, I would mention the chorale by Johann Sebastian Bach which was placed at the end of the *Kunst der Fuge* (1749). (Figure 12) The chorale melody, slightly ornamented, is in the upper voice. The other voices prepare for the entry of the chorale by imitating the melody in diminution (twice as fast). The alto voice plays the inversion of the melody, and most of the other accompanying material is derived directly from the opening measures. The entire piece consists of three more phrases, all treated in the same way. Here self-similarity consists of the intensely repeated use of the same motives within one larger section of the piece.

The self-similarity of this piece (or the economy of the motives, to speak in more usual musical terms) is quite astonishing. But it is important to understand that such a degree of self-similarity is exceptional in traditional music. Such intensive use of motives together with such complex and strict contrapuntal writing always creates musical tension and drama. This is, so to speak, not musics natural state. The music gives clear evidence of the creative force which was able to shape it in such a fashion. (The chorale is a late composition. The title: "I herewith stand before Thy throne" and the fact that Bach twice signs his name numerologically in the chorale melody itself indicate that the piece had special significance for him. The tension and drama to which the economy of motive and the strict counterpoint give rise emphasize and express this significance.)

Figure 12. Partial self-similarity in the chorale prelude "Vor deinen Thron tret' ich hiermit" BWV 668. The lower three voices imitate the chorale in diminution, the alto in inversion. Brackets indicate other correspondences.

The self-similarity of the Bach choral is remarkable, but it is only partial and it concerns only the motive material of the piece, not its harmony or its form. In particular, except for the diminution of the three lower voices, scaling invariance is almost completely absent. More radical examples of self-similarity and scaling invariance can be found in the twentieth century. An important example is the Concerto for Nine Instruments op. 24 by Anton Webern. (Figure 13)

Figure 13. The Concerto for Nine Instruments *op. 24 by Anton Webern, first movement, mm. 1-8. The tonal material is strongly self-similar, but scaling invariance is not important in the work as a whole.*

In the first nine measures the instruments play only three-note figures consisting of the same two intervals, major third (or minor sixth) and minor ninth (or major seventh) in one of four different speeds. Especially in the first five measures, where the four speeds appear equally often, the basic tempo of the music (the scaling) is unclear. The form of this movement is more traditional and is based on the sonata-allegro form, although here too Webern creates a form which is much more highly symmetrical than the classical form. But while the three-tone motive can be found in every measure, thus ensuring strong self-similarity of the tonal material, scaling invariance is not very important in the work as a whole. The tempo becomes clear in the sixth measure, and the music flows on, leading our perception forward. Individual phrases are grouped to larger units by means of dynamics, tempo and other rather traditional means. But these groups are very differently constructed from the three-note motive themselves, and the movements form is yet again different. We shall return to this problem of scalability later.

Serial music (after 1950) offers the most radical examples of self-similarity. In the field of electroacoustic music, there is considerable documentation showing to what degree the early music of Karlheinz Stockhausen (*Elektronische Studie II*, *Gesang der Jnglinge* and *Kontakte*) uses the same (numerical) elements to construct the sounds themselves, build phrases and derive formal structure. Elsewhere (*Proceedings II of the International Academy of Electroacoustic Music 1996*, Bourges/Paris) I have described some of the self-similar structures of my piece *Rainstick*.

In all these examples, but particularly in the electroacoustic pieces mentioned, it is important to point out that self-similarity and scaling invariance really only apply to the generating numbers and proportions. To claim, as I do in my analysis of *Rainstick*, that there is a musical relation between sounds whose partials are ordered according to a set of proportions and temporal ordering of the sounds using the same set of proportions, is speculative at best. Because the materials which are being ordered differ so greatly from each other, it is difficult to speak of meaningful self-similarity, let along to argue that such self-similarity is of the slightest musical relevance. This is very different from the visual representation of the Mandelbrot Set on a computer screen, where the physical position of a point in the complex plane determines its belonging or not to the set, and where the definition of a "dwell band" corresponds to a mathematical characteristic of that point (namely, how many reiterations of the "Mandelbrot formula" were necessary to cause the calculated value to surpass 2.0).

At the beginning of this text I pointed out self-similarity and scaling invariance as the two central features of "chaotic" systems. The examples above showed that neither of these features is particularly characteristic of traditional music (except perhaps in trivial, non-compositional ways: the similarity in structure between the partials of harmonic sounds and the triads of traditional harmony, or the division of the beat into shorter, equally long parts—the quarter note into sixteenths, for example—and the combination of the beat to larger units–the quarter into regular measures, for example), and this fact is reason enough to reflect briefly on the reasons why self-similarity and scaling invariance appear relatively rarely in music.

Self-similarity as repetition of motives is of course not particularly rare. Interest in economy of material is a characteristic of certain historical styles and thus has varied over the centuries, but composers have frequently chosen to weave the fabric of their music from the same few elements. To be sure, the powerful constraints of harmony and counterpoint in traditional music made it difficult to imagine the next-larger structure, the phrase, being handled in the same way as the motive, and it is almost unthinkable that large-scale entities, groups of two or three minutes duration, be treated in the same way as motives. But what about electroacoustic music today, where such constraints no longer exist? I shall try to imagine iterating the same operations on several levels of material from my piece

Rainstick in the manner of the formulas we have seen to generate self-similar, non-scalable patterns, in the hope of learning something about the appropriateness of such structures for music.

The smallest compositional element in *Rainstick* is the sound, either the result of synthesis of or transformation of a recorded sound. Most of the sounds in the piece have spectra whose energy is distributed according to a simple set of eight proportions (that is, either partials appear at frequencies having those relationships to each other, or resonances occur at those frequencies). At the next level, individual sounds are both transposed and ordered in time (both in individual duration and in time of entry) according to the same set of proportions. Let me consider this transposition and the ordering in time as the two basic operations performed on my basic material. There are many ways to imagine the application of these two operations at the next highest level. I shall choose the most obvious: transpose the original as many times as there are sounds, then choose the duration and starting time of each resulting phrase according to the basic proportions. Repeat the process for each new resulting unit as often as desired.

How self-similar is the resulting music? I spoke above about how speculative it is to consider many different materials self-similar, just because they are ordered using the same numbers. In our example at least the material is always the same. But the partials I group compositionally change their basic nature when they fuse in the perception to become a sound. And the phrase of nine sounds is fundamentally different from a single sound. If I reiterate the process often enough to obtain a result having a duration measured in minutes, that result will yet again be absolutely different in character from the original, no longer just a complicated phrase, but a formal process. Our perception that the sound, the phrase and the formal unit are fundamentally different from one another has to do with our perception of time. The synchronicity of the partials attacks and decay fuse them into one sound. The succession of events over a time-span so long that we no longer group them into one perceptual unit gives rise to the sense of form. Phrases are longer than the individual sound but shorter than a formal unit.

What about scaling invariance? Does it obtain throughout our imaginary example? No, and for the same reason. Time distinguishes quite precisely between a sound and a succession of sounds, less precisely but usually quite efficiently between a succession of sounds and a formal structure. There is at least one well-known physical limit which produces this essential change of percept: below about 25 Hz we hear individual pulses of amplitude and see individual images, above this limit, we hear tones and see continuous movement. Other perceptual limits doubtless have many components, memory among them, but change of perceived quality (is it a sound? is it a phrase?) as a function of time is an important trait of the acoustical perception.

So we see that neither self-similarity nor scaling invariance seems very robust in music. The beauty we perceive in the Mandelbrot Set surely has something to do with the order (and disorder) in the generating equations, but we perceive this beauty because we see the Set all at once, outside of time. Even if we gave each point in the complex plane musical significance, playing the individual points of the Mandelbrot Set line by line would hardly be as interesting as looking at the visual representation. It would seem that self-similarity and scaling invariance apply to music only over short temporal intervals. When these characteristics appear, as in Bach, Webern or Stockhausen, their presence indicates heightened significance and is accompanied by great technical tension. But the flow of time and the functioning of the auditory perception mitigate against self-similarity and scaling invariances playing an central role in musical composition.

And so at the end of my very cursory exploration of chaos, I can understand that as mechanisms for engendering sounds, streams of sounds, phrases, techniques related to chaotic systems may be of considerable interest to composers today. But because of the way sonic events are perceived in time, it is difficult for me to imagine that the essential nature of chaotic systems—self-similarity and scaling invariance—could ever be of real structural importance to music.

One of the very finest books about chaos, Manfred Schroeder's *Fractals, Chaos and Power Laws* [10] has the subtitle: "Minutes from an Infinite Paradise". Here, I thought, is a partial explanation of my scepticism towards Chaos Theory in music: if the music of the twentieth century has taught us anything, it is that there is no paradise about which to write. Even if it is possible to express the beauty of chaotic systems not just visually, but also aurally, this beauty could hardly be the subject of serious music today. Mathematical chaos, when treated with insight, reveals astonishing beauty, a beauty whose regularity derives from the strictly deterministic techniques employed to give birth to it. Music, on the other hand, must deal not with number, but with real, sounding, materials. When treated with insight, these too reveal great beauty, rougher, less regular than fractal beauty, to be sure, but beauty within which the echoes of the real Chaos can clearly be heard.

1 Appendix

Here are two computer programs to illustrate the logistic difference function. The first is a very simple program in C which writes the values of the function between two values for the growth function (here called r). The second is a program for the sound synthesis language Csound showing simple ways to make audible the logistic difference function.

```
/*
bifurcate.c
A very simple program illustrating the logistic difference function.
*/
```

```
#include <stdio.h>
#include <string.h>

#define NUMBER_OF_ITERATIONS 1000

main()
{
int i, j, num_of_values;
float r, x, start, end, increment;
FILE *fp;
char name;

printf("File name for function file: ");
scanf("%s", &name);
if ( (fp = fopen(&name, "w")) == NULL) {
printf("Couldn't open file %s\n", &name);
exit();
}
printf("Start value for r (growth value): ");
scanf("%f", &start);
printf("End value for r (growth value): ");
scanf("%f", &end);
printf("How many values: ");
scanf("%d", &num_of_values);

increment = (end - start) / (float) (num_of_values-1);

r = start;
x = 0.5;

fprintf(fp, "Logistic difference function\n");
fprintf(fp, "%d values between %f and %f\n\n", num_of_values, start, end);
fprintf(fp, "    r\t\t     x\n\n");

for (j=0; j < num_of_values-1; j++)
{
for (i=1; i < NUMBER_OF_ITERATIONS; i++)
x *= r * (1-x);
fprintf(fp, "%f\t%f\n", r, x);
r += increment;
```

```
}
r = end;
for (i=1; i < NUMBER_OF_ITERATIONS; i++)
x *= r * (1-x);
fprintf(fp, "%f\t%f\n", r, x);

fclose(fp);

}
```

```
; bifurcate.orc
; a Csound orchestra demonstrating very
; simple applications of the logistic difference
; function in sound synthesis

sr=44100
kr=10 ;  the control rate detmines
; how many pitches are played by instrument 2
ksmps=4410
nchnls=1

instr 1

; This instrument creates the waveform directly.
; A duration of 0.743 seconds gives 32768 samples.
; The logistic difference equation is used to calculate
; the amplitude directly.

kterm phasor 1/p3

ireiterate = 100

; p4 starting value for r
; p5 ending value for r

istart  = p4
iend  = p5
iextent = iend - istart

krvar = kterm * iextent + istart
```

```
kcounter = 1
kx = 0.5

jumpback:
if (kcounter == ireiterate) kgoto jump
kx = kx * krvar * (1.0 - kx)
kcounter  = kcounter + 1
kgoto jumpback

jump:

kx = (kx * 2) - 1.0
a1 = kx * 32767

out a1

endin

instr 2

; This instrument produces a tone whose pitch
; is controlled by the logistic difference function
; The value of the control rate (kr) in the header
; determines how often a new pitch is chose;
; One can choose any portion of the logistic
; difference function by setting p5 and p6 appropriately

kterm phasor 1/p3

ireiterate = 100


; p4 amplitude
; p5 starting value for r, the growth factor
; p6 ending value for r
; p7 higest pitch
; p8 lowest pitch

iamp =  p4
istart  =  p5
iend  =  p6
```

```
iextent =  iend - istart

irange = p7 - p8

krvar = kterm * iextent + istart
kcounter = 1
kx = 0.5

jumpback:
if (kcounter == ireiterate) kgoto jump
kx = kx * krvar * (1.0 - kx)
kcounter  = kcounter + 1
kgoto jumpback

jump:
kpitch = kx * irange
a1 oscili iamp, kpitch, 1

out a1

endin

instr 3

; This instrument is a variant of instrument 2.
; The only difference is that it calculates
; the pitch logarithmically.

kterm phasor 1/p3

ireiterate = 100

; p4 amplitude
; p5 starting value for r, the growth factor
; p6 ending value for r
; p7 higest pitch
; p8 lowest pitch

iamp =  p4
istart  =  p5
iend  =  p6
```

```
iextent =  iend - istart

ilogp1 = log(p6)
ilogp2 = log(p7)
idiff = ilogp2 - ilogp1

krvar = kterm * iextent + istart
kcounter = 1
kx = 0.5

jumpback:
if (kcounter == ireiterate) kgoto jump
kx = kx * krvar * (1.0 - kx)
kcounter  = kcounter + 1
kgoto jumpback

jump:
klogp = kx * idiff + ilogp1
kpitch = exp(klogp)
a1     oscili 20000, kpitch, 1

out a1

endin
--------------------------------------------------

; bifurcate.sco

f1 0 32768 10 1

; instr 1:
; p4 starting value for r
; p5 ending value for r

; instr 2 and instr 3:
; p4 amplitude
; p5 starting value for r, the growth factor
; p6 ending value for r
; p7 higest pitch
; p8 lowest pitch
```

```
; p4 p5 p5 p6 p7

; example notes for each instrument

;i1 0 0.743 3.831 3.8317
i2 0 4 20000 3.7 3.8 500 125
;i3 0 4 20000 3.7 3.8 500 125
e
```

In the original version of this text, written in 1996, I included a short bibliography of books and articles about chaos and fractals in general ([2], [8], [9]) and more specifically about chaotic structures in music ([1], [3], [5], [11]). Of the publications since 1996, I would like to call particular attention to the book by Martin Neukom [7], whic includes computer programs and sound examples.

In the original version I also included numerous Internet addresses , where a great deal of information was and is available. In the meantime, Internet searches yield literally millions of references, and it hardly seems necessary to list a selection here.

References

[1] Jean-Pierre Boon and Olivier Docroly. Dynamical systems theory for music dynamics. *Chaos*, 5(3):501–508, 1995.

[2] Heinz-Otto Peitgen et al. *Chaos and Fractals: New Frontiers of Science*. Springer Verlag, New York, 1992.

[3] Martin Gardner. Mathematical games. *Scientific American*, April, 1978:16–32, 1978.

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